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Review of the Numerical Analysis of the Quadratic Riccati Equation using Adomian Decomposition Methods and Taylor Expansion

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Abstract
Riccati differential equations are a class of non-linear differential equations with numerous physical applications. In this research, we described a numerical method based on decomposition and compared the results to the exact solution. Riccati's differential equations have quadratic solutions expressed as an endless string using an iterative approach is introduced. The solutions of Riccati differential equations in the quadratic form are exhibited in terms of an infinite series, which are obtained by the iterative algorithm. We conducted comparisons between the exact and approximate solutions. The similarities between the Taylor series approach and the new method highlight the latter is simplicity and efficacy. We used tables and graphs by using MATLAB to show how similar the current method is to the exact solution.

Keywords: Riccati**,** differential equations**,** an infinite series**,** MATLAB**.**

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مراجعة التحليل العددي لمعادلة ريكاتي التربيعية باستخدام طريقتي التحلل لأدوميان ومتسلسلة تايلور

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الملخص

معادلات ريكاتي التفاضلية هي فئة من المعادلات التفاضلية غير الخطية ذات التطبيقات الفيزيائية العديدة. في هذا البحث قمنا بوصف طريقة عددية تعتمد على التحلل ومقارنة النتائج بالحل الدقيق. تحتوي معادلات ريكاتي التفاضلية على حلول تربيعية يتم التعبير عنها بسلسلة لا نهاية لها باستخدام نهج تكراري. يتم عرض حلول معادلات ريكاتي التفاضلية في الصورة التربيعية بدلالة سلسلة لا نهائية، والتي تم الحصول عليها عن طريق الخوارزمية التكرارية. لقد أجرينا مقارنات بين الحلول الدقيقة والتقريبية وإن أوجه التشابه بين نهج سلسلة تايلور وطريقة التحلل لأدوميان تسلط الضوء على بساطة الأخيرة وفعاليتها. لقد استخدمنا الجداول والرسوم البيانية باستخدام برنامج MATLAB لإظهار مدى تشابه الطريقة الحالية مع الحل الدقيق.

الكلمات المفتاحية: ريكاتي، المعادلات التفاضلية، المتسلسلة اللانهائية، MATLAB.

1.Introduction

The Italian mathematician Jacopo Francesco Riccati formulated the Riccati differential equation [1], the first-order non-linear ordinary differential equation is a prevalent element in various branches of mathematics different and is utilized in engineering, science, physics, and many applications. Analytically, certain types of differential equations pose challenges in terms of finding efficient and explicit solutions using straightforward methods. In 1988, George Adomian introduced the Adomian Decomposition Method (ADM) [2],itcan yield an approximation solution to Riccati

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differential equation, as can the Taylor expansion technique [3]. These approaches were used to solve a variety of linear and non-linear equations, and the results were close to precise.
The subject has received significant attention and research by many different researchers. Several methods have been employed to solve quadratic Riccati differential equations, including the Bezier curve method [4], the multistage variable iteration method [5], and Legendre scaling functions, which expand the approximate solution as Legendre elements.

In this study, we solve the Riccati differential equation by using the Adomian Decomposition Method and the Taylor expansion method, and give two examples during which the approximate the solution was compared to the exact solution and the error was determined. Graphics were also presented to illustrate this. The results indicate that a small number of terms are sufficient for a rapid convergence of an approximate solution that accurately reflects the actual solutions.

Definition1.1 [6, 1]
Riccati equation is a differential equation of the form:

$$y'(x) + a(x)y(x) + b(x)y^2(x) + c(x) = 0\,, y(0) = d \qquad (1)$$

Where d is a constant, $y(x)$ is an unknown function, $a(x), b(x)$ and $c(x)$ are functions of x.

2. Analysis of Methods
2.1. Adomian Decomposition Method: [7, 8, 9]
A differential equation can be written as:

$$Fy = g, \qquad (2)$$

Where F is a non-linear differential operator with linear and non-linear terms.The linear part of F is decomposed as

$$F = L + R \;, \qquad (3)$$

Where L is an invertible operator and R is a linear operator's residual. Using L as the highest-order derivative simplifies the equation to:

$$Ly + Ry + Ny = g\,, \quad (4)$$

Where Ny corresponds to the non-linear terms. Rewrite (4), we get:

$$Ly = g - Ry - Ny\,, \quad (5)$$

Applying the operator L^{-1} to the equation (5), we obtain

$$L^{-1}(Ly) = L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny)\,. \quad (6)$$

Suppose h is the solution of the homogeneous equation $Ly = 0$, with the given initial/boundary conditions. Then the general solution of equation (6) is:

$$y(x) = h + L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny). \quad (7)$$

The next problem is the decomposition of the non-linear term Ny. Adomian developed a very elegant method, the approximate solution of equation (7) can be written as:

$$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots = \sum_{n=0}^{\infty} \lambda^n y_n(x)\,, \quad (8)$$

Where λ is constant, whereas $y_0(x), y_1(x), y_2(x), \dots, y_n(x)$ are sought. If the non-linear operator $N$is attempted to equation (8):

$$Ny = N(y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots).$$

The non-linear term Ny will be decomposed by the infinite series of Adomian:

$$Ny = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots = \sum_{n=0}^{\infty} \lambda^n A_n, \quad (9)$$

With:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n} \lambda^k y_k(x)\right)\right]\,, \quad (10)$$

Where A_n are called Adomian polynomials, if$N(y) = f(y)$, the Adomian polynomials are given as:

$$A_0 = f(y_0)$$
$$A_1 = y_1 f'(y_0)$$

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$$A_2 = \frac{y_1}{2!} f''(y_0) + y_2 f'(y_0)$$
$$A_3 = \frac{{y_1}^3}{2!} f'''(y_0) + y_1 y_2 f''(y_0) + y_3 f'(y_0)$$
$$\vdots$$

Suppose $L^{-1}Ry$, $L^{-1}Ny$ have a λ order, then Equation (7) can be written as:

$$y(x) = h + L^{-1}g - \lambda L^{-1}(Ry) - \lambda L^{-1}(Ny). \tag{11}$$

Put equation (8) and (9) in equation (11), then we obtain:

$$\sum_{n=0}^{\infty} \lambda^n y_n(x) = h + L^{-1}g - \lambda L^{-1}\left(R\left(\sum_{n=0}^{\infty} \lambda^n y_n\,(x)\right)\right) - \lambda L^{-1}\left(\sum_{n=0}^{\infty} \lambda^n A_n\right). \tag{12}$$

Equating the coefficients of equal powers λ on both sides of equation (12), we obtain:

$$y_0 = h + L^{-1}g$$
$$y_1 = -L^{-1}\,R(y_0) - L^{-1}(A_0)$$
$$y_2 = -L^{-1}\,R(y_1) - L^{-1}(A_1)$$
$$\vdots$$

In general can be expressed by the recursive relations

$$y_n = -L^{-1}\,R(y_{n-1}) - L^{-1}(A_{n-1}), \qquad n \geq 1. \tag{13}$$

Then, the approximate solution is given by

$$y = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots \ . \tag{14}$$

2.2. Taylor Expansion Method:

Taylor's Theorem 2.1. [3] Suppose y is continuous on the closed interval [a,b] and has $n+1$ continuous derivatives on open interval

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(a, b).If x and x_0 are points in (a, b),then the Taylor series expansion of $y(x)$ about x_0 :

$$y(x_0) + y'(x_0)(x - x_0) + \frac{(x - x_0)^2}{2!} y^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} y^{(3)}(x_0) + \cdots$$

Or

$$y(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} y^{(n)}(x_0) \ , \qquad n, x_0 \ \in (a, b), \tag{15}$$

Where $y(x)$is convergence.

2.2.1. Convergence Analysis

Theorem 2.2.[10] Let N be a non-linear operator from Hilbert space $\mathbb{H}$ where: $\mathbb{H} \longrightarrow \mathbb{H}$ and y be the exact solution of (14). The decomposition series $\sum_{n=0}^{\infty} y_n$ of y converges to y when $\exists\ \alpha < 1$, $||y_{n+1}|| \leq \alpha ||y_n||, \forall n \in N \cup \{0\}$.

3. Applications and Numerical Results [9]

Method is a reliable approach that requires less work compared to traditional procedures. To provide a clear understanding, here are some examples of the methods that will be explained. Matlab program is utilized to compute all the results.

Example 3.1

Consider the quadratic Riccati differential equation is:

$$y' = -y^2 + 1 \ , \qquad y(0) = 0, \tag{16}$$

Where the exact solution is:

$$y = \frac{e^{2x} - 1}{e^{2x} + 1} \,. \tag{17}$$

We need to solve equation (16) using the Adomian decomposition approach and the Taylor expansion method, and then compare it to the exact solution (17).

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Method 1: Using Adomian Decomposition Method

From equation (16) we get $c(x) = 1, and\ \ b(x) = -1\ , a(x) = 0.$ We find$y_0, y_1, y_2, \ldots$,from Adomian polynomial (13) can be derived as follows at $n \geq 1$,:

$$y_0 = x\ ,$$
$$y_1 = -\frac{x^3}{3}\ ,$$
$$y_2 = \frac{2}{15}x^5,$$
$$y_3 = -\frac{17}{315}x^7\ ,$$
$$y_4 = \frac{62}{2835}x^9\ ,$$
$$\vdots$$
$$y_{n+1} = y_0 + y_1 + y_2 + y_3 + y_4 + \cdots + \ \ y_n\ ,$$

From equation (14) we get:

$$y = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots. \qquad (18)$$

Method 2: Using Taylor expansion series

$$y = x - \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{x^7}{126} + \cdots, \qquad (19)$$

We use Matlab programming to solve them, find exact and approximate solutions, and compare them.

In table1 and figure 1, the results we obtained using the mentioned methods are presented and compared with the complete solution, and the error amount is calculated.

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```
function riccati1
symsy
y=0:0.1:1
uapp=(y-(y.^3)/3+(2/15)*y.^5-(17/315)*y.^7)
%+(62/2835)*y.^9)
%uex=(exp(2.*y)-1.)/(exp(2*y)+1.)
uex=tanh(y)
taylor=y-y.^3/3+(2/15)*y.^5-y.^7/126
erorr=abs((uapp-uex))
erortaylor=abs((taylor-uex))
```

TABLE1. Computed Approximate and Exact Solution for Example (3.1)

x	Exact solution	APS (ADM)	Error |ADM-Exact|	APS(TS)	Error |TS-Exact|
0	0	0	0	0	0
0.1	0.0997	0.0997	0	0.0997	0
0.2	0.1974	0.1974	0	0.1974	0
0.3	0.2913	0.2913	0	0.2913	0
0.4	0.3799	0.3799	0	0.38	0
0.5	0.4621	0.4621	0	0.4624	0
0.6	0.5370	0.5371	0	0.5381	2E-4
0.7	0.6044	0.6045	1E-4	0.6074	7E-4
0.8	0.6640	0.6646	6E-4	0.6714	23E-4
0.9	0.7163	0.7184	21E-4	0.7319	64E-4
1	0.7616	0.7679	63E-4	0.7921	15E-3

*APS(ADM)= Approximate solution (ADM) & APS(TS)= Approximate Solution (Taylor Series) .

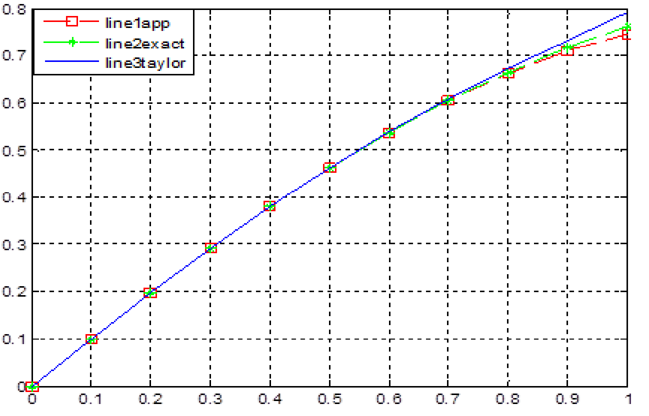

Figure 1: Comparison of Approximate Solution and Exact Solution for Example (3.1).

Example 3.2 Consider the first order Riccati differential equation

$$y' = y^2 - 2xy + x^2 + 1\,, \quad y(0) = \frac{1}{2}\,. \tag{20}$$

The exact solution is:

$$y = \frac{1}{2-x} + x. \tag{21}$$

We solve equation (20) using the Adomian decomposition approach and the Taylor expansion method, then compare them to the exact solution (21) in the table 2. Figure 2 shows the Comparison of Approximate Solution and Exact Solution for the example.

TABLE 2. Computed Approximate and Exact Solution for Example (3.2)

x	Exact solution	APS(ADM)	Error |ADM-Exact|	APS(TS)	Error |TS-Exact|
0	0.5	0.5	0	0.5	0
0.1	0.6263	0.6263	0	0.6262	1E-4
0.2	0.7556	0.755	6E-4	0.7547	11E04
0.3	0.8882	0.8864	22E-4	0.8851	3E-3
0.4	1.025	1.0205	5E-3	1.0172	12E-3
0.5	1.1667	1.1577	11E-3	1.1507	16E-3
0.6	1.3143	1.2984	15E-3	1.2853	17E-2
0.7	1.4692	1.4433	26E-3	1.4207	4E-2

0.8	1.6333	1.5931	39E-3	1.5565	12E-2
0.9	1.8091	1.7482	6E-2	1.6925	29E-2
1	2	1.9091	9E-2	1.8281	2E-1

*APS(ADM)= Approximate Solution (ADM) & APS(TS)= Approximate Solution (Taylor Series).

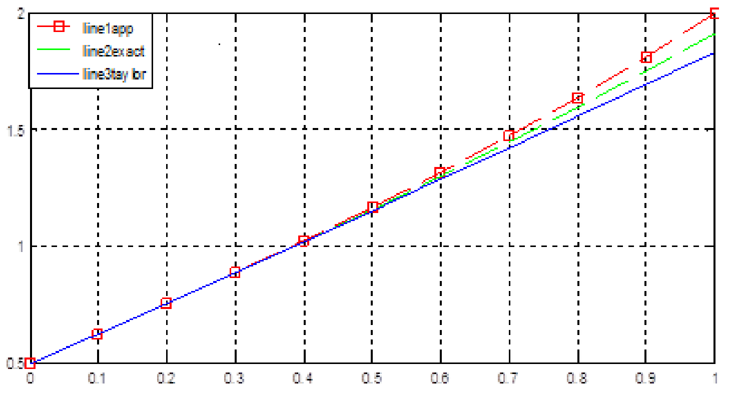

Figure 2: Comparison of Approximate Solution and Exact Solution for Example (3.2).

4. Conclusion

In this study, we present a numerical methodology for solving Riccati differential equations of the quadratic type. The numerical results in the tables and figures show that determining the approximate solution of the Riccati differential equation of quadratic type is effective and clear. We compared the Adomian decomposition method and Taylor expansion approaches for solving the Riccati equation. Adomian decomposition method and exact solutions produce results that are very similar. Adomian decomposition method is a trustworthy and successful strategy. The effectiveness gives it a much wider appropriateness, which has to be explored more.

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